Note on two results on the rainbow connection number of graphs*

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Abstract

An edge-colored graph G, where adjacent edges may be colored the same, is rainbow connected if any two vertices of G are connected by a path whose edges have distinct colors. The rainbow connection number $rc(G)$ of a connected graph G is the smallest number of colors that are needed in order to make G rainbow connected. Caro et al. showed an upper bound $rc(G) \leq n - \delta$ for a connected graph G of order n with minimum degree δ in "On rainbow connection, Electron. J. Combin. 15(2008), R57". Recently, Shiermeyer gave it a generalization that $rc(G) \leq n - \frac{\sigma_2}{2}$ in "Bounds for the rainbow connection number of graphs, Discuss. Math Graph Theory 31(2011), 387–395", where σ_2 is the minimum degree-sum. The proofs of both results are almost the same, both fix the minimum degree δ and then use induction on n. This short note points out that this proof technique does not work rigorously. Fortunately, Caro et al's result is still true but under our improved proof. However, we do not know if Shiermeyer's result still hold.

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We follow the notation and terminology in [1], as well as [2] and [3]. The parameter *degree-sum* $\sigma_2(G)$, or simply σ_2, is defined as $min\{d(u) + d(v) | u, v \in V(G)\ and\ uv \notin E(G)\}$.

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1 The problem in the proof of Proposition 2.5 of [2]

In [2], Caro et al. got an upper bound of the rainbow connection number in terms of the minimum degree δ of a connected graph, which is stated as follows:

Proposition 1.1 (Proposition 2.5 of [2])**.** *If G is a connected graph with minimum degree δ, then $rc(G) \leq n - \delta$.*

In the proof they claimed that they will fix δ and prove the proposition by induction on n. But the fact is that δ_i of G_i is always different from δ. So we cannot use the induction hypothesis. Fortunately, the proposition is true. We give an improved proof of the proposition by induction on $n - \delta$.

Proof. We prove the proposition by induction on $n - \delta$. The base case $n - \delta = 1$ is trivial since cliques have rainbow connection number $1 = n - \delta$. Assume that the proposition is true for all connected graphs with $n - \delta < s$. Let us consider a connected graph with $n - \delta = s$.

Let K be a maximal clique of G consisting only of vertices whose degree is δ. Since there is at least one vertex with degree δ and since G is connected we have $1 \leq k = |K| \leq \delta$.

Consider the graph G' obtained from G by deleting the vertices of K. Suppose the connected components of G' are $G_1, \ldots, G_t$ where G_i has n_i vertices and minimum degree δ_i for $i = 1, \ldots, t$. Let $K_i \subseteq K$ be the vertices of K with a neighbor in G_i, and assume that $|K_1| \geq |K_i|$ for $i = 1, \ldots, t$ (notice that it may be that $t = 1$ and G' is connected). Consider first the case that $K_1 = K$. We know that $n - n_i \geq |K| = k$. For any vertex v in G_i, if $d_G(v) > \delta$, then $d_{G_i}(v) \geq d_G(v) - k > \delta - k$; if $d_G(v) = \delta$, then v can not be adjacent to every vertex in K by the maximality of K, so $d_{G_i}(v) \geq \delta - (k-1) > \delta - k$. Thus we have $n - n_i \geq k > \delta - \delta_i$, namely $n_i - \delta_i < n - \delta = s$. By the induction hypothesis, $rc(G_i) \leq n_i - \delta_i$. Clearly, we may give the edges of K and the edges from K to G_1 the same color. Hence,

$$\begin{aligned} rc(G) &\leq t + \sum_{i=1}^{t}(n_i - \delta_i) = t + n - k - \sum_{i=1}^{t}\delta_i \\ &\leq t + n - k - t(\delta - k + 1) \\ &= t + n - k - t\delta + tk - t \\ &= n - \delta - (t-1)(\delta - k) \leq n - \delta. \end{aligned}$$

Now assume that $K_1 \neq K$ but that $|K_1| = k_1 > 1$. Similarly, we have $n_i - \delta_i < n - \delta = s$. If there exists a G_i with $\delta_i \geq \delta - k + 2$, then we can give the edges of K a new color.

Hence,

$$\begin{aligned}
rc(G) &\leq t+1+\sum_{i=1}^{t}(n_i-\delta_i) = t+1+n-k-\sum_{i=1}^{t}\delta_i \\
&\leq t+1+n-k-(t-1)(\delta-k+1)-(\delta-k+2) \\
&= t+1+n-k-(t-1)(\delta-k)-t+1-\delta+k-2 \\
&= n-\delta-(t-1)(\delta-k) \leq n-\delta.
\end{aligned}$$

Otherwise, every G_i has $\delta_i = \delta-k+1$, namely there is a vertex v_i in G_i adjacent to at least $k-1$ vertices in K for $i = 1,\ldots,t$. Since $K_1 \neq K$, there is a vertex $u \in K$ but $u \notin K_1$. For any v_i, $i = 1,\ldots,t$, v_i is not adjacent to at most 1 vertex in K. $|K_1| = k_1 > 1$, so v_i is adjacent to at least 1 vertex in K_1. Now consider any two vertices v_i and v_j, $1 \leq i < j \leq t$. If v_i and v_j are both adjacent to u, then they have a common neighbor in K; if one of them, say v_i is not adjacent to u, then v_i is adjacent to all vertices in K_1, so they also have a common neighbor in K. Hence we may give the edges of K a used color, and so $rc(G) \leq t + \sum_{i=1}^{t}(n_i-\delta_i) \leq n-\delta$.

Finally, if $k_1 = 1$ (and since $K_1 \neq K$ we have $k \geq 2$), contract all of K into a single vertex v. Notice that the contracted graph G^* has $n-k+1$ vertices and $\delta^* \geq \delta$. So $n-k+1-\delta^* < n-\delta = s$. By induction hypothesis, $rc(G^*) \leq n-k+1-\delta^* \leq n-k+1-\delta$. Going back to G and coloring the edges of the clique K with another new color, we obtain $rc(G) \leq n-k-\delta+2 \leq n-\delta$. The proof is now complete. ∎

2 The problem in the proof of Theorem 9 of [3]

As a generalization of the above proposition, Schiermeyer got the following upper bound of the rainbow connection number:

Proposition 2.1 (Theorem 9 of [3]). *If G is a connected graph with minimum degree-degree σ_2, then $rc(G) \leq n - \frac{\sigma_2}{2}$.*

In the proof of the result, Schiermeyer used the same proof method as in Proposition 2.5 of [2]. So the proof is also not correct. Besides, the author claimed that "each pair of nonadjacent vertices of G_i has degree-sum at least $\sigma_2(G)-2(k-1)$ in G_i", this is not correct. We can give a counter-example. Just take t copies of $K_{\delta+4}$ as $H_1,\ldots,H_t$, $t < \frac{\delta+1}{2}$. $v_{i,1}$ and $v_{i,2}$ are two nonadjacent vertices, $i = 1,\ldots,t$. Connect $v_{i,1}$ and $v_{i,2}$ to any $2t$ vertices in H_i, respectively, the neighbor of $v_{i,1}$ and $v_{i,2}$ can be same or different. Let $K = K_{\delta-2t+1}$ and connect $v_{i,1}$, $v_{i,2}$ to all vertices in K. Denote the graph by G. Then the vertices in H_i have degree at least $\delta+3$, $v_{i,1}$, $v_{i,2}$ have degree $\delta+1$ and the vertices in K have

degree δ. Since every vertex with degree δ is adjacent to every vertex with degree at most $\delta+1$ and other vertices have degree at least $\delta+3$, $\sigma_2(G)=d(v_{i,1})+d(v_{i,2})=2(\delta+1)$. In $G_i=G[H_i\cup\{v_{i,1},v_{i,2}\}]$, $v_{i,1}$ and $v_{i,2}$ are two nonadjacent vertices and $d_{G_i}(v_{i,1})+d_{G_i}(v_{i,2})=4t$. But $\sigma_2(G)-2(k-1)=2(\delta+1)-2(\delta-2t+1-1)=4t+2>4t=d_{G_i}(v_{i,1})+d_{G_i}(v_{i,2})$, contradicting the claim. On the other hand, since $d_{G_i}(v)\geq d_G(v)-k$, we can get $s_i\geq \sigma_2(G)-2k$. Therefore

$$\begin{aligned}
rc(G) &\leq t+\sum_{i=1}^{t}(n_i-\frac{s_i}{2})=t+n-k-\sum_{i=1}^{t}\frac{s_i}{2}\\
&\leq t+n-k-t(\frac{\sigma_2(G)}{2}-k)\\
&=t+n-k-t\frac{\sigma_2(G)}{2}+tk\\
&=n-\frac{\sigma_2(G)}{2}-(t-1)(\frac{\sigma_2(G)}{2}-k)+t\\
&\leq n-\frac{\sigma_2(G)}{2}+t.
\end{aligned}$$

When t is large, this bound is far from $n-\frac{\sigma_2(G)}{2}$ given by the author of [3]. However, we do not know whether his theorem is still true.

References

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